

# Eigenvalue Asymptotics of Perturbed Self-adjoint Operators

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**Abstract.** We study perturbations of a self-adjoint positive operator  $T$ , provided that a perturbation operator  $B$  satisfies "local" subordinate condition  $\|B\varphi_k\| \leq b\mu_k^\beta$  with some  $\beta < 1$  and  $b > 0$ . Here  $\{\varphi_k\}_{k=1}^\infty$  is an orthonormal system of the eigenvectors of the operator  $T$  corresponding to the eigenvalues  $\{\mu_k\}_{k=1}^\infty$ . We introduce the concept of  $\alpha$ -non-condensing sequence and prove the theorem on the comparison of the eigenvalue-counting functions of the operators  $T$  and  $T + B$ . Namely, it is shown that if  $\{\mu_k\}$  is  $\alpha$ -non-condensing then the difference of the eigenvalue-counting functions is subject to relation

$$|n(r, T) - n(r, T + B)| \leq C [n(r + ar^\gamma, T) - n(r - ar^\gamma, T)] + C_1$$

with some constants  $C, C_1, a$  and  $\gamma = \max(0, \beta, 2\beta + \alpha - 1) \in [0, 1)$ .

The results of this paper are published in [17].

## 1. Introduction

Throughout this paper,  $T$  shall stand for a self-adjoint and bounded below operator with domain  $\mathcal{D}(T)$  acting in a separable Hilbert space  $\mathcal{H}$ . We always suppose that  $T$  has a discrete spectrum which is denoted by  $\{\mu_k\}_{k=1}^\infty$  and each eigenvalue  $\mu_k$  is repeated in the sequence in accordance with its geometric multiplicity. A complete orthonormal system of the eigenvectors that correspond to these eigenvalues we denote by  $\{\varphi_k\}_{k=1}^\infty$ . For convenience we always assume that  $1 < \mu_k \leq \mu_{k+1}$  for all integers  $k \geq 1$ . Let

$$(1.1) \quad n(r, T) = \sum_{\mu_k < r} 1$$

be the eigenvalue-counting function of the operator  $T$ . We suppose that there is a positive number  $\alpha$ , such that

$$(1.2) \quad \overline{\lim}_{r \rightarrow \infty} \frac{n(r, T)}{r^\alpha} = C < \infty.$$

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This condition is natural as it is fulfilled for a large class of differential operators (ordinary and with partial derivatives on a bounded domain in  $\mathbb{R}^n$ , see [14], for example). In the case  $\alpha = 1$  we say that a sequence  $\{\mu_k\}_{k=1}^\infty$  is *non-condensing* if there is a number  $l$  such that each segment  $(t, t+1]$ ,  $t \in \mathbb{R}^+$ , contains at most  $l$  eigenvalues of the operator  $T$ . Obviously, this condition is equivalent to the following: *for all  $t > 1$  the inequality  $n(t+0, T) - n(t-1, T) \leq l$  holds*. In the case  $\alpha \neq 1$  we introduce the concept of  *$\alpha$ -non-condensing sequence*. Namely, we say that a sequence  $\{\mu_k\}_{k=1}^\infty$  satisfying the condition (1.2) is  *$\alpha$ -non-condensing* if a sequence  $\{\mu_k^\alpha\}_{k=1}^\infty$  is non-condensing, or equivalently  $n(t^{1/\alpha} + 0, T) - n((t-1)^{1/\alpha}, T) \leq l$  with some  $l \in \mathbb{N}$ .

The goal of this paper is to obtain results on the distribution of the eigenvalues for the perturbations  $A = T + B$ , provided that the perturbation operator  $B$  satisfies the conditions

$$(1.3) \quad \mathcal{D}(B) \supset \mathcal{D}(T), \quad \|B\varphi_k\| \leq b\mu_k^\beta, \quad -\infty < \beta < 1,$$

where  $b$  is a constant. The main our result is as follows.

**THEOREM 1.** *Let conditions (1.2) and (1.3) be fulfilled and the sequence  $\{\mu_k\}_{k=1}^\infty$  be  $\alpha$ -non-condensing. Assume that*

$$(1.4) \quad \gamma = \max(0, \beta, 2\beta + \alpha - 1) < 1.$$

*Then the spectrum of the operator  $A = T + B$  consists of isolated eigenvalues  $\{\lambda_k\}$  and there exist positive constants  $a, C$  and  $C_1$  such that the eigenvalue-counting function*

$$n(r, A) = \sum_{|\lambda_k| < r} 1$$

*is subject to the relation*

$$(1.5) \quad |n(r, A) - n(r, T)| \leq CS_\gamma(r) + C_1,$$

*where*

$$S_\gamma(r) = n(r + ar^\gamma, T) - n(r - ar^\gamma, T).$$

Viewing in mind applications, it is worth mentioning the following corollary.

**COROLLARY 1.** *Assume that*

$$n(r, T) = r^\alpha + O(r^\eta), \quad \text{with some } \eta < \alpha.$$

*Then under assumptions of Theorem 1 we have*

$$|n(r, A) - n(r, T)| = O(r^{\alpha+\gamma-1}) + O(r^\eta)$$

The proof is obvious.

The asymptotic behavior of the eigenvalues of self-adjoint operators and their perturbations has long been studied by many authors. Asymptotic formulae for the Sturm-Liouville operator were found in 19-th century. The first general results on the eigenvalue distribution of the eigenvalues of ordinary differential operators were obtained by Birkhoff [4], and for partial differential operators by Weyl [18]. Keldysh [8] proved the first result for relatively compact perturbations of general self-adjoint operators in Hilbert space using Tauberian technique. The most complete and sharp results for compact perturbations and for the so-called  $\beta$ -subordinate perturbations of self-adjoint operators are due to Markus and Matsaev [11] (see more details in [10, Ch.1]. Additional information on

eigenvalue distribution of self-adjoint operators and their perturbations can be found in the book of Naimark [12], the survey of Rosenblum, Solomyak and Shubin [14], the book of Markus [10], the survey of Agranovich [3].

Here we remark that the main novelty of our paper is the subordinate condition (1.3). The sharpest results on the comparison of spectra of original and perturbed operators which are due to Markus and Matsaev [11], dealt with subordinate conditions of the form

$$(1.6) \quad \|Bf\| \leq C\|T^\beta f\| \quad \forall f \in \mathcal{D}(B) \supset \mathcal{D}(T^\beta)$$

or

$$(1.7) \quad \|Bf\| \leq C\|Tf\|^\beta \|f\|^{1-\beta} \quad \forall f \in \mathcal{D}(T).$$

Here  $\beta < 1$  and  $C$  is a constant independent on  $f$ . We also note that the second condition here is weaker than the first one. Obviously, our condition (1.3) is essentially weaker than conditions (1.7). We shall demonstrate this by a simple example.

Consider the self-adjoint operator  $T = i\frac{d}{dx}$  on the domain

$$\mathcal{D}(T) = \{f : f \in W_2^1(0, 2\pi), f(0) = f(2\pi)\}$$

in the Hilbert space  $\mathcal{H} = L_2[0, 2\pi]$ . The eigenvalues of  $T$  are equal  $\mu_k = k, k \in \mathbb{Z}$ , and the eigenfunctions coincide with trigonometric system  $\varphi_k = \{e^{ikx}\}_{k=-\infty}^{\infty}$ . Then, consider as a perturbation the multiplication operator  $Bf = b(x)f(x)$ , where  $b(x) \in L_2(0, 2\pi)$  and has sufficiently strong singularity at some point  $x_0 \in [0, 2\pi]$ , say  $b(x) = \frac{1}{\ln(x/4\pi)\sqrt{x}}$ . Then  $\|B\varphi_k\| \leq \|b(x)\|$ , i.e. the condition (1.3) holds with  $\beta = 0$ . It is known [5] that  $\mathcal{D}(|T|^\beta) = W_2^\beta$  for any  $0 \leq \beta < 1/2$ , where  $W_2^\beta = W_2^\beta(0, 2\pi)$  is the Sobolev space with smooth index  $\beta$ . It can easily be verified that the function  $f_0(x) = \ln(x/4\pi)$  belongs to the space  $W_2^\beta$  for any  $\beta < 1/2$ . Hence, for these values of  $\beta$  we have  $Bf_0 \notin L_2(0, 2\pi)$ , while  $|T|^\beta f_0 \in L_2(0, 2\pi)$ . Therefore, condition (1.6) is not fulfilled for any  $\beta < 1/2$ . Then, the same is true for condition (1.7), because the validity of (1.7) with some  $\beta < 1$  implies the validity of (1.6) with any  $\beta' < \beta$  (see [10, Ch 1], for example). This example shows that in particular situations the subordinate condition (1.3) can be much more effective than (1.6) or (1.7). Simultaneously we have to say that Theorem 1 does not generalize the Markus -Matsaev theorem [11]. Condition (1.6) or (1.7) implies the validity of Theorem 1 with the function

$$S(r) = n(r + ar^\beta, T) - n(r - ar^\beta, T),$$

i.e.  $\gamma$  can be replaced by  $\beta$ . Therefore, assuming a weaker assumption (1.3) instead of (1.6) or (1.7), we get the same estimate as in the Markus-Matsaev theorem only in the case  $\beta \leq 2\beta + \alpha - 1$ , i.e.  $\beta + \alpha \leq 1$ . Otherwise, we have to pay for a weaker assumption getting estimate (1.5) which is less sharp.

Finally, we remark that "local" subordinate condition (1.3) was originated in author's paper [15], where Theorem 1 was proved for the case  $\alpha = 1$  and  $\beta = 0$ .

## 2. Proof of Theorem 1

First, we shall prove Theorem 1 for the case  $\alpha = 1$ . In this case the proof is more transparent and technically much easier. Later on we will explain how to overcome the technicalities in the case  $\alpha \neq 1$ . While proving this result we will use the trick of "artificial lacuna" proposed by Markus and Matsaev in [11]. However, the implementation of this trick will be organized in a technically different way. Our plan to prove the theorem

is the following. First, we prove relation (1.4) for a fixed  $r$ , provided that the interval  $(r - 2ar^{2\beta}, r + 2ar^{2\beta})$  does not contain the eigenvalues of the operator  $T$ , where  $a$  is a certain number depending on the numbers  $b$  and  $\beta$  and independent of  $r$ . For such  $r$  we show the equality  $n(r, A) = n(r, T)$ . Then, for each fixed  $r$  we construct a finite rank self-adjoint operator  $K_r$  commuting with  $T$  such that

- (i) the operator  $T_r = T - K_r$  has no eigenvalues in the interval  $(r - ar^{2\beta}, r + ar^{2\beta})$ ;
- (ii) the property (1.3) remains valid for  $T - K_r$  with the constant  $2b$  instead of  $b$ ;
- (iii) The inequality  $|n(r, T) - n(r, T_r)| \leq S_\gamma(r)$  holds.

Then we apply the Weinstein-Aronszajn formula from the theory of perturbation determinants (see [7, Ch.5])

$$(2.1) \quad n(r, A) = n(r, T - K_r + B) + \nu(r, h),$$

where  $\nu(r, h)$  denotes the difference between the numbers of zeros and poles of the scalar meromorphic function  $D(\lambda) := \det(1 - K_r(\lambda - T + K_r - B)^{-1})$ , that lie in the rectangle  $\mathcal{R}$  which is bounded by vertical lines  $\operatorname{Re} \lambda = r$ ,  $\operatorname{Re} \lambda = -R$  and the horizontal lines  $\operatorname{Im} \lambda = \pm R$  with sufficiently large  $R = R(r)$ . We remark that formula (2.1) can easily be proved by using the identity

$$1 - K_r(\lambda - T + K_r - B)^{-1} = (\lambda - A)(\lambda - T + K_r - B)^{-1}.$$

Since the operator  $T_r = T - K_r$  has no eigenvalues in the interval  $(r - 2ar^{2\beta}, r + 2ar^{2\beta})$  we get  $n(r, T_r + B) = n(r, T_r)$ . On the other hand, by construction we have  $|n(r, T_r) - n(r, T)| \leq N = S_\gamma(r)$ . Therefore, we will prove (1.5) if we show that the function  $|\nu(r, h)|$  is bounded by the right hand-side of (1.5). To show the latter assertion is the main technical difficulty in the proof of Theorem 1 which we shall divide into several steps.

*Step 1.* We will use in the sequel the following result from complex analysis.

LEMMA 2. *Let  $f$  be a function that is bounded and analytic in the rectangle*

$$(2.2) \quad \Pi = \{\lambda : |\operatorname{Re} \lambda - r| < c, |\operatorname{Im} \lambda| < d\}.$$

*For  $\delta \in (0, 1)$ , set  $c' = c(1 - \delta)$  and  $d' = d(1 - \delta)$ , and denote by  $\Pi'$  the rectangle defined by (2.2) where  $c$  and  $d$  are replaced by  $c'$  and  $d'$ . Denote*

$$M = \sup_{\lambda \in \Pi} f(\lambda), \quad M' = \sup_{\lambda \in \Pi'} f(\lambda).$$

*Then there is a constant  $C$  depending on  $\delta$  and the ration  $c/d$  and independent of  $f$  such that the following holds:*

- (i) *The number  $n_f(\Pi')$  of zeros of the function  $f$  inside the rectangle  $\Pi'$  is subject to the estimate*

$$(2.3) \quad n_f(\Pi') \leq C(\ln M - \ln M').$$

- (ii) *If  $\gamma$  is a straight line segment contained in  $\Pi'$  that does not pass through the zeros of  $f$  in  $\Pi'$ , then the variation of the argument of the function  $f$  along  $\gamma$  is subject to the same estimate*

$$(2.4) \quad |[\arg f(\lambda)]_\gamma| \leq C(\ln M - \ln M').$$

PROOF. Some versions of this assertion can be found in the monograph [9, Ch. 1] and in [16]. In the form presented here this result is contained in [11, Lemmas 1.1 and 1.3].  $\square$

*Step 2.*

LEMMA 3. *Under assumptions of Theorem 1 the operator  $A = T + B$  has discrete spectrum.*

PROOF. Since  $T$  is self-adjoint the following representation for the resolvent holds

$$(2.5) \quad (\lambda - T)^{-1} = \sum_{k=1}^{\infty} \frac{(\cdot, \varphi_k) \varphi_k}{\lambda - \mu_k}.$$

Without loss of generality we have assumed that the point  $\lambda = 0$  belongs to the resolvent set of  $T$ . Denoting  $f_k = (f, \varphi_k)$  and taking into account that  $\sum |f_k|^2 = \|f\|^2$  we get

$$(2.6) \quad \left\| BT^{-1}f - \sum_{k=1}^N \frac{f_k B \varphi_k}{\mu_k} \right\| = \left\| \sum_{k=N+1}^{\infty} \frac{f_k B \varphi_k}{\mu_k} \right\| \leq \|f\|^2 \sum_{k=N+1}^{\infty} \frac{\|B \varphi_k\|^2}{\mu_k^2} \\ \leq b^2 \|f\|^2 \sum_{k=N+1}^{\infty} \mu_k^{2\beta-2} \leq \varepsilon \|f\|^2,$$

where  $\varepsilon \rightarrow 0$  as  $N \rightarrow \infty$ . The latter assertion holds since condition (1.2) with  $\alpha = 1$  implies  $\mu_k \geq C^{-1}k$ , hence, the series  $\sum \mu_k^{2\beta-2} \leq C^{2-2\beta} \sum k^{2\beta-2}$  converges (here we use our assumption (1.4) which implies  $\beta < 1/2$ , provided that  $\alpha = 1$ ). The estimate (2.6) shows that  $BT^{-1}$  is compact, therefore  $B$  is a relatively compact perturbation of  $T$ . Then the discreteness of the spectrum of  $T + B$  follows from lemma of Keldysh (see, for example, [10, Lemma 3.6]).  $\square$

*Step 3.*

LEMMA 4. *Let the spectrum  $\{\mu_k\}_1^{\infty}$  of the operator  $T$  form a non-condensing sequence. Equivalently, there is a number  $l \in \mathbb{N}$  such that*

$$(2.7) \quad n(t+0) - n(t-1) \leq l \quad \text{for all } t \geq 1.$$

*Then there is a continuous piece-wise linear function  $\psi(t)$  such that*

$$|n(t) - \psi(t)| \leq l$$

*and*

$$|\psi'(t)| \leq l.$$

PROOF. Without loss of generality we have already assumed that  $\mu_1 > 1$ . Define the integers  $s_m := n(m+0)$ . Then the segments  $\Delta_m = (m-1, m]$ ,  $m = 1, 2, \dots$ , contain  $l_m = s_m - s_{m-1} \leq l$  eigenvalues from the sequence  $\{\mu_k\}_{k=1}^{\infty}$ . Now define the function  $\psi(t)$  on the interval  $\Delta_{m+1} = (m, m+1]$  as follows

$$\psi(t) = s_m + l_m(t - m), \quad t \in (m, m+1].$$

It follows from the construction that  $|\psi(t) - n(t)| \leq \sup\{l_m\} = l$  and  $\psi'(t) \leq l$ . The lemma is proved.  $\square$

*Step 4.* Let us prove the following lemma.

LEMMA 5. Let  $a$  be a fixed positive number and suppose that the interval  $(r - 2ar^{2\beta}, r + 2ar^{2\beta})$  does not contain the points  $\mu_k$  of the spectrum of the operator  $T$ . Assume also that the constant  $b$  participating in condition (1.3) is such that

$$(2.8) \quad a \geq 48lb^2.$$

Then the following estimate is valid in the strip  $r - ar^{2\beta} \leq \operatorname{Re} \lambda \leq r + ar^{2\beta}$ :

$$(2.9) \quad \sum_{k=1}^{\infty} \frac{\|B\varphi_k\|^2}{|\lambda - \mu_k|^2} < \frac{1}{4},$$

provided that  $r \geq C$  where  $C = C(a, \beta)$  depends only on  $a$  and  $\beta$  (hence, only on  $l, b$  and  $\beta$ ).

PROOF. Denote  $\lambda = \operatorname{Re} \lambda + i \operatorname{Im} \lambda := \sigma + i\tau$  and  $r^- = r - 2ar^{2\beta}$ ,  $r^+ = r + 2ar^{2\beta}$ . We have to estimate the sum for the values  $\sigma \in (r - r^{2\beta}, r + r^{2\beta})$  and  $\tau \in \mathbb{R}$ . Recall that according to Lemma 4 we have the representation

$$n(t) = \psi(t) + \zeta(t), \quad \text{where } |\psi'(t)| \leq l, \quad |\zeta(t)| \leq l, \quad n(t) = n(t, T).$$

Using condition (1.3) we obtain

$$(2.10) \quad \sum_{k=1}^{\infty} \frac{\|B\varphi_k\|^2}{|\lambda - \mu_k|^2} \leq b^2 \sum_{k=1}^{\infty} \frac{\mu_k^{2\beta}}{(\sigma - \mu_k)^2 + \tau^2} =$$

$$b^2 \int_1^{\infty} \frac{t^{2\beta} dn(t)}{(\sigma - t)^2 + \tau^2} \leq b^2 l \left( \int_1^{r^-} + \int_{r^+}^{\infty} \right) \frac{t^{2\beta} dt}{(\sigma - t)^2 + \tau^2} +$$

$$b^2 \left( \int_1^{r^-} + \int_{r^+}^{\infty} \right) \frac{(2\beta t^{2\beta-1} [(\sigma - t)^2 + \tau^2] + 2t^{2\beta} |\sigma - t|) |\zeta(t)| dt}{[(\sigma - t)^2 + \tau^2]^2}.$$

Remark that we have integrated by parts while transforming here the integrals. Taking into account the inequalities

$$|\sigma - t| / \sqrt{(\sigma - t)^2 + \tau^2} \leq 1, \quad 2\beta < 1, \quad |\zeta(\xi)| \leq l,$$

we estimate the last sum of the integrals as follows

$$(2.11) \quad \leq l \left( \int_1^{r^-} + \int_{r^+}^{\infty} \right) \left[ \frac{2\beta t^{2\beta-1}}{(\sigma - t)^2 + \tau^2} + \frac{2t^{2\beta}}{((\sigma - t)^2 + \tau^2)^{3/2}} \right] dt \leq$$

$$3l \left( \int_1^{r^-} + \int_{r^+}^{\infty} \right) \frac{t^{2\beta} dt}{(\sigma - t)^2 + \tau^2} \leq 3l \left( \int_1^{r^-} + \int_{r^+}^{\infty} \right) \frac{t^{2\beta} dt}{(\sigma - t)^2},$$

provided that  $\min[(\sigma - r^-), (r^+ - \sigma)] = ar^{2\beta} \geq 1$ . Therefore, to prove lemma it is sufficient to estimate the integral

$$(2.12) \quad \left( \int_1^{r^-} + \int_{r^+}^{\infty} \right) \frac{t^{2\beta} dt}{(\sigma - t)^2} \leq \int_{\sigma-r^-}^{\sigma} \frac{(\sigma - \xi)^{2\beta}}{\xi^2} d\xi + \int_{r^+-\sigma}^{\infty} \frac{(\xi + \sigma)^{2\beta}}{\xi^2} d\xi \leq$$

$$\sigma^{2\beta} \int_{ar^{2\beta}}^{\infty} \frac{1}{\xi^2} d\xi + \int_{ar^{2\beta}}^{\sigma} \frac{(2\sigma)^{2\beta}}{\xi^2} d\xi + \int_{\sigma}^{\infty} \frac{(2\xi)^{2\beta}}{\xi^2} d\xi < \frac{(1 + 2^{2\beta})\sigma^{2\beta}}{ar^{2\beta}} + \frac{2^{2\beta}}{(1 - 2^{2\beta})} \sigma^{2\beta-1} <$$

$$\frac{1 + 2^{2\beta}}{a} (1 + ar^{2\beta-1})^{2\beta} + \frac{2^{2\beta}}{(1 - 2^{2\beta})} (r - ar^{2\beta})^{2\beta-1} < \frac{3}{a} - \frac{(2 - 2^{2\beta})}{a} + Cr^{2\beta-1} < \frac{3}{a},$$

provided that  $r$  is sufficiently large, i.e.  $r^{1-2\beta} \geq Ca/(2 - 2^{2\beta})$  where  $C$  depends only on  $\beta$  and  $a$ . Now, the assertion of lemma straightly follows from (2.10), (2.11) and (2.12). The Lemma is proved.  $\square$

*Step 5.* Here we estimate the left hand-side of (2.9) outside the parabolic domain defined as follows

$$(2.13) \quad P(h, 2\beta) = \{\lambda : |Im\lambda| \leq h(Re\lambda)^{2\beta}, Re\lambda \geq 0\}$$

LEMMA 6. *Let the conditions of Theorem 1 hold. Let  $h$  be a positive number and*

$$\sigma_h = \left[ \frac{2h}{\pi(1 - 2\beta)(2^{1-2\beta} - 1)} \right]^{1/(1-2\beta)}.$$

*Then for all  $\lambda = \sigma + i\tau$  lying in the half-plane  $\sigma \geq \sigma_h$  and outside the parabola  $P(h, 2\beta)$  defined by (2.13) the following estimate holds*

$$(2.14) \quad \sum_{k=1}^{\infty} \frac{\|B\varphi_k\|^2}{|\lambda - \mu_k|^2} < \frac{6\pi b^2 l}{h}.$$

PROOF. Repeating the arguments in proving Lemma 5 (see estimates (2.10) and (2.11)) we get

$$(2.15) \quad \sum_{k=1}^{\infty} \frac{\|B\varphi_k\|^2}{|\lambda - \mu_k|^2} < 4b^2 l \int_1^{\infty} \frac{t^{2\beta} dt}{(\sigma - t)^2 + \tau^2}.$$

Further we proceed

$$\int_1^{\infty} \frac{t^{2\beta} dt}{(\sigma - t)^2 + \tau^2} < \int_0^{\sigma} \frac{(\sigma - \xi)^{2\beta}}{\xi^2 + \tau^2} d\xi + \left( \int_0^{\sigma} + \int_{\sigma}^{\infty} \right) \frac{(\sigma + \xi)^{2\beta}}{\xi^2 + \tau^2} d\xi <$$

$$\sigma^{2\beta} \frac{\pi}{2\tau} + (2\sigma)^{2\beta} \frac{\pi}{2\tau} + \int_{\sigma}^{\infty} \frac{(2\xi)^{2\beta}}{\xi^2} d\xi.$$

Outside the parabola  $P_h$  we have  $\tau > h\sigma^{2\beta}$ . Therefore, for  $\lambda = \sigma + i\tau \notin P_h$  the right hand-side of the last inequality can be estimated as follows

$$< \frac{3\pi}{2h} - \frac{(2 - 2^{2\beta})\pi}{2h} + \frac{2^{2\beta}}{(1 - 2^{2\beta})} \sigma^{2\beta-1} \leq \frac{3\pi}{2h},$$

provided that  $\sigma \geq \sigma_h$ . This ends the proof of lemma.  $\square$

*Step 6.*

LEMMA 7. *Let conditions of Lemma 5 hold. Then there is a large number  $R = R(r)$  such that the estimate (2.9) holds on the boundary of the rectangle  $\mathcal{R}_r$  whose horizontal sides are the segments of the lines  $\operatorname{Im} \lambda = \pm R$  and the vertical sides are the segments of the lines  $\operatorname{Re} \lambda = -R$  and  $\operatorname{Re} \lambda = r$ .*

PROOF. The validity of estimate (2.9) on the line  $\operatorname{Re} \lambda = r$  is proved in Lemma 5. It follows from the proof of Lemma 6 that the left hand-side of (2.9) obeys the estimate  $\leq C(\operatorname{Im} \lambda)^{-1}$  as  $\operatorname{Im} \lambda \rightarrow \infty$  uniformly in the half-plane  $\operatorname{Re} \lambda < \sigma_h$ . We do not show here details because they are obviously seen from the proof of Lemma 6. Finally, the estimate on the line  $\operatorname{Re} \lambda = -R$  with sufficiently large  $R$  also follows easily from the above representations.  $\square$

Step 7. Our goal at this step is to show the equality  $n(r, T_r + B) = n(r, T_r)$ , where  $T_r$  is a special finite-dimensional "correction" of the unperturbed operator  $T$ . First we construct the operator  $T_r$ .

Take a positive  $a$  such that the inequality

$$(2.16) \quad a \geq 96 b^2 l$$

holds. We pay attention that this condition differs from (2.8) by changing  $l$  to  $2l$  (we will use this further). Fix a number  $r$  such that  $r - 2ar^{2\beta} > 1$ . Define the interval  $\Delta_r = (r - 2ar^{2\beta}, r + 2ar^{2\beta})$  and the operator

$$K_r = 4ar^{2\beta} \sum_{\mu_k \in \Delta_r} (\cdot, \varphi_k) \varphi_k.$$

Obviously,  $K_r$  is a self-adjoint operator of finite rank not exceeding the value

$$(2.17) \quad N = n(r + 2ar^{2\beta}, T) - n(r - 2ar^{2\beta}, T)$$

Now define  $T_r = T + K_r$ . Obviously, this operator preserves the system of eigenfunctions  $\{\varphi_k\}_1^\infty$  but changes the eigenvalues lying in the interval  $\Delta_r = (r - 2ar^{2\beta}, r + 2ar^{2\beta})$ ; it shifts them by  $4ar^{2\beta}$  to the right from this interval. The condition  $T_r \geq 1$  is preserved, since  $K_r$  is non-negative. The sequence of the eigenvalues  $\{\mu'_k\}_{k=1}^\infty$  of the operator  $T_r$  remains non-condensing but we have to take into account that the number  $l = \sup_{t>0} \left( \sum_{\mu_k \in [t, t+1)} 1 \right)$  is changed to  $2l$ . Then, by construction  $\mu'_k \geq \mu_k$ , therefore condition (1.3) is preserved for  $T_r$ .

Let us estimate the norm of the operator function  $B(\lambda - T_r)^{-1}$  in the strip  $\operatorname{Re} \lambda \in (r - ar^{2\beta}, r + ar^{2\beta})$ . We apply the method used by Adduci and Mityagin [1, §4] or [2]. Let  $f \in \mathcal{H}$ ,  $\|f\| = 1$ , and  $f_k = (f, \varphi_k)$  be the Fourier coefficients of the element  $f$ . Then

$$(2.18) \quad \|B(\lambda - T_r)^{-1} f\|^2 = \left\| \sum_{k=1}^{\infty} \frac{f_k B \varphi_k}{\lambda - \mu'_k} \right\|^2 \leq \sum_{k=1}^{\infty} |f_k|^2 \sum_{k=1}^{\infty} \frac{\|B \varphi_k\|^2}{|\lambda - \mu'_k|^2} = \sum_{k=1}^{\infty} \frac{\|B \varphi_k\|^2}{|\lambda - \mu'_k|^2}.$$

Applying Lemma 5 and taking into account that the number  $a$  is selected by (2.16) instead of (2.8) (because the number  $l$  for  $T_r$  has to be changed by  $2l$ ), we obtain the following estimate

$$(2.19) \quad \|B(\lambda - T_r)^{-1}\| < \sqrt{\frac{1}{4}} = \frac{1}{2}, \quad \operatorname{Re} \lambda \in (r - ar^{2\beta}, r + ar^{2\beta}).$$



Now, by virtue of Lemma 7 take a number  $R = R(r)$  such that estimate (2.19) holds on the boundary of the rectangle  $\mathcal{R}$ . Then, for all  $t \in [0, 1]$  the Riesz projectors

$$Q_t = \frac{1}{2\pi i} \int_{\partial\Omega} (\lambda - T_r - tB)^{-1} d\lambda = \frac{1}{2\pi i} \int_{\partial\Omega} (\lambda - T_r)^{-1} (1 - tB(\lambda - T_r)^{-1}) d\lambda$$

are well defined and depend continuously on  $t$  in the norm operator topology. By virtue of Szökefalvi-Nagy's lemma (see [6, Ch. 1, Lemma 3.1])  $\dim Q_t = \dim Q_\xi$ , provided that  $\|Q_t - Q_\xi\| < 1$ . Therefore, it follows from the continuity of  $Q_t$  that

$$(2.20) \quad n(r, T_r) = \dim P_0 = \dim P_1 = n(r, T_r + B).$$

This proves lemma.

*Step 8.* Consider the scalar function

$$D(\lambda) = \det(1 - K_r(\lambda - T_r - B)^{-1}).$$

By virtue of Lemma 3 the spectrum of the operator  $T_r + B$  is discrete. Hence, the operator function  $K(\lambda) := K_r(\lambda - T_r - B)^{-1}$  is meromorphic and its values are finite rank operators for  $\lambda \notin \sigma(T_r + B)$ . Therefore, the determinant  $D(\lambda)$  is well defined (as a meromorphic function) and is equal to the product  $\prod_j (1 - \lambda_j(K))$ , where  $\lambda_j(K)$  are the eigenvalues of the operator  $K(\lambda)$ . Since  $\dim K_r \leq 4ap$ , this product contains at most  $4ar^{2\beta}$  factors.

LEMMA 8. *In the strip  $\operatorname{Re} \lambda \in (r - ar^{2\beta}, r + ar^{2\beta})$  the function  $D(\lambda)$  is holomorphic and is estimated as*

$$(2.21) \quad |D(\lambda)| \leq 9^N,$$

where the number  $N$  is defined by (2.17). At the point  $\lambda = r + ihr^{2\beta}$  the following lower estimate holds

$$(2.22) \quad |D(\lambda)| \geq \left(\frac{1}{2}\right)^N, \quad \text{provided that } h \geq 16a.$$

PROOF. We shall use the identity

$$(\lambda - T_r - B)^{-1} = (\lambda - T_r)^{-1} (1 - B(\lambda - T_r)^{-1})^{-1}$$

and the estimates

$$\|(\lambda - T_r)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(T_r))} \leq \frac{1}{ar^{2\beta}}, \quad \|(1 - B(\lambda - T_r)^{-1})^{-1}\| \leq 2$$

which hold for  $\lambda$  in the strip  $\operatorname{Re} \lambda \in (r - ar^{2\beta}, r + ar^{2\beta})$ . The first estimate here is valid because  $T_r$  is self-adjoint, and the second one is proved in Lemma 7. In particular, we find that the operator function  $K(\lambda)$  is holomorphic in the strip  $|\operatorname{Re} \lambda - r| < ar^{2\beta}$  and its eigenvalues obey the inequalities

$$|\lambda_j(K)| \leq \|K_r\| \|(\lambda - T_r - B)^{-1}\| \leq 4ar^{2\beta} \cdot \frac{1}{ar^{2\beta}} \cdot 2 = 8.$$

The number of the eigenvalues is equal to the rank of the operator  $K(\lambda)$ , which does not exceed the rank of the operator  $K_r$  equal to  $N$ . Therefore, the product of  $N$  factors of the form  $(1 - \lambda_j(K))$  is subject to estimate (2.21).

Next, by virtue of Lemma 5 the estimate  $\|B(\lambda - T_r)^{-1}\| < 1/2$  is valid for  $\lambda$  on the line  $\operatorname{Re} \lambda = r$ . Using the resolvent estimate for the self-adjoint operator  $T_r$  we get for  $\lambda = r + i h r^{2\beta}$

$$(2.23) \quad |\lambda_j(K(\lambda))| < \|K_r\| \|(\lambda - T_r)^{-1}\| \|(1 - B(\lambda - T_r)^{-1})^{-1}\| \leq 4a r^{2\beta} \cdot \frac{1}{r^{2\beta} \sqrt{1 + h^2}} \cdot 2 < \frac{8a}{h} < 1/2,$$

provided that  $h \geq 16a$ . Then  $|1 - \lambda_j(K)| > 1/2$ . Therefore, the product of  $N$  factors of this form can be estimated as follows

$$|D(\lambda)| \geq \left(1 - \frac{1}{2}\right)^N \geq \left(\frac{1}{2}\right)^N, \quad |\operatorname{Im} \lambda| \geq h r^{2\beta} \geq 16 a r^{2\beta}.$$

The lemma is proved.  $\square$

*Step 9.* Now we are ready to prove the main lemma.

LEMMA 9. *Fix numbers  $a \geq 48 b^2 l$  and  $h \geq 16a$ . Take sufficiently large  $r$  and consider a rectangle  $\mathcal{R}$  defined in Lemma 5 with  $R > 2hr^{2\beta}$ . Suppose that the line  $\operatorname{Re} \lambda = r$  does not contain the eigenvalues of the operator  $A$ . Then the variation of the argument along the boundary of the rectangle  $\mathcal{R}$  is subject to the estimate*

$$|[\arg D(\lambda)]|_{\partial \mathcal{R}} \leq CN + C_1,$$

where  $N$  is defined by (2.17) and  $C, C_1$  are constants depending only on  $l$  and  $b$ .

PROOF. We have proved in the previous lemma estimate (2.23) for all  $\lambda \in \partial \mathcal{R}$  except the segment on the line  $\operatorname{Re} \lambda = r$  with the endpoints  $r \pm i h r^{2\beta}$ . Hence, the variation of the argument of the functions  $(1 - \lambda_j(K(\lambda)))$  when  $\lambda$  varies along  $\partial \mathcal{R}$  between these points outside this segment does not exceed  $\pi/3$ . Then the variation of the argument of the function  $D$  along this curve is  $\leq \pi N/3$ .

To complete the proof we have to estimate the variation of the argument along the segment  $I_r = [r - i h r^{2\beta}, r + i h r^{2\beta}]$ . For this purpose we shall use Lemma 2. First, we chose a number  $a$  satisfying condition (2.16). Then, we take a number  $h$ , say,  $h = 16 a b^2$  such that inequality (2.22) holds. Assume that  $r \geq C$  where  $C$  is the constant from Lemma 5. Consider the rectangle  $R_{a,h}$  bounded by the straight lines  $\operatorname{Re} \lambda = r \pm a, \operatorname{Im} \lambda = \pm 2h$  and denote by  $R'_{a,h}$  the twice contracted rectangle with the same center at the point  $r$ . Lemma 2 together with estimates (2.21) and (2.22) imply that the variation of the argument of the function  $D$  along the segment  $I_r$  (provided that this segment does not pass through the zeros of the function  $D$ ) does not exceed  $C'(\ln 9 + \ln 2)N$  where  $C'$  is an absolute constant. This proves the lemma.  $\square$

*Step 10.* It follows from Lemma 9 that the difference between the number of zeros and poles of the function  $D$  inside the rectangle  $\mathcal{R} = \mathcal{R}_r$  does not exceed  $CN \leq C'(b^2 l)N$  where  $C'$  is an absolute constant. Note also that by construction of  $T_r$  we have  $0 \leq n(r, T) - n(r, T_r) \leq N$ . Therefore, formula (2.1) gives  $|n(r, A) - n(r, T)| \leq CN$ , provided that  $r \geq C_0$ . Taking  $C_1 = n(C_0, T)$  we get the assertion of Theorem 1. We have only to explain what to do with exceptional values of  $r$  when the segment  $I_r$  passes through the zeros of the function  $D$  which coincide with the eigenvalues of  $A$ . To explain this we remark that all these zeros of the function  $D$  lie in the rectangle  $R'(a, h)$  and

by virtue of Lemma 2 the number of these zeros is  $\leq CN$ . Therefore, the jump of the function  $n(r, A)$  does not exceed this value, and the relation (1.5) remains valid for all  $r \in \mathbb{R}^+$ . This ends the proof of Theorem 1 for the case  $\alpha = 1$ .

*Step 11.* Let  $\alpha > 0$  and  $\gamma := 2\beta + \alpha - 1$ ,  $0 \leq \gamma < 1$ . An important step in the proof of the theorem for the case  $\alpha = 1$  was made in Lemma 5. In the general case we also have the estimate

$$(2.24) \quad \sum_{k=1}^{\infty} \frac{\|B\varphi_k\|^2}{|\lambda - \mu_k|^2} \leq b^2 \sum_{k=1}^{\infty} \frac{\mu_k^{2\beta}}{(\sigma - \mu_k)^2 + \tau^2} = b^2 \int_1^{\infty} \frac{t^{2\beta} dn(t)}{(\sigma - t)^2 + \tau^2}.$$

Since the sequence  $\{\mu_k\}_{k=1}^{\infty}$  is  $\alpha$ -non-condensing, the function  $n(\xi^{1/\alpha}) = n(\xi^{1/\alpha}, T)$  by virtue of Lemma 4 can be represented in the form

$$n(\xi^{1/\alpha}) = \psi(\xi) + \zeta(\xi), \quad 0 \leq \psi'(\xi) \leq l, \quad |\zeta(\xi)| \leq l.$$

Denote  $\lambda = \sigma + i\tau$ ,  $r^- = r - 2ar^\gamma$ ,  $r^+ = r + 2ar^\gamma$  and assume that this interval does not contain the eigenvalues of the operator  $T$ . Then, we can rewrite the integral in the right hand-side of (2.24) as follows

$$(2.25) \quad \int_1^{\infty} \frac{\xi^{2\beta/\alpha} d[\psi(\xi) + \zeta(\xi)]}{(\sigma - \xi^{1/\alpha})^2 + \tau^2} \leq l \int_1^{\infty} \frac{\xi^{2\beta/\alpha} d\xi}{(\sigma - \xi^{1/\alpha})^2 + \tau^2} + \int_1^{\infty} \left| \left[ \frac{\xi^{2\beta/\alpha}}{(\sigma - \xi^{1/\alpha})^2 + \tau^2} \right]' \right| |\zeta(\xi)| d\xi.$$

The second integral in the right hand-side of (2.25) obeys the estimate

$$(2.26) \quad \leq l \int_1^{\infty} \left[ \frac{2\beta t^{2\beta-1}}{(\sigma - t)^2 + \tau^2} + \frac{2t^{2\beta}}{[(\sigma - t)^2 + \tau^2]^{3/2}} \right] dt.$$

Recalling that  $\mu_k \notin (r^-, r^+)$  we can replace the integral  $\int_1^{\infty}$  by  $\int_1^{r^-} + \int_{r^+}^{\infty}$ . Then, the left hand-side in (2.25) is subject to estimate (for all  $\tau \in \mathbb{R}$ )

$$(2.27) \quad \leq \alpha l \left( \int_1^{r^-} + \int_{r^+}^{\infty} \right) \left[ \frac{t^\gamma}{(\sigma - t)^2 + \tau^2} + \frac{2\beta t^{2\beta-1}}{(\sigma - t)^2 + \tau^2} + \frac{2t^{2\beta}}{(\sigma - t)^3} \right] dt$$

Here the integral from the first summand can be estimated in the same way as in Lemma 5, namely, it is  $\leq \frac{(1+2\gamma)}{a} + Cr^{\gamma-1}$ . There are no problems with the estimation of the integral from the second summand because  $2\beta - 1 < \gamma$ . Finally, let us estimate, for example, the first integral from the third summand. We have

$$\int_1^{r^-} \frac{t^{2\beta}}{(\sigma - t)^3} dt \leq \sigma^{2\beta} \int_{ar^\gamma}^{\sigma^{-1}} \frac{1}{x^3} dx \leq \frac{\sigma^{2\beta}}{a^2 r^{2\gamma}} \leq \frac{2}{a^2},$$

provided that  $r > C = C(a, \beta)$  and  $\beta \leq \gamma$ . In the case  $\beta > \gamma = 2\beta + \alpha - 1$  we have to put  $\gamma = \beta$ . Then the last estimate holds. The previous estimates for the first and the second summand in (2.27) are also valid because  $2\beta + \alpha - 1 < \beta$ .

Therefore, we have proved that the assertion of Lemma 5 remains valid in the general case  $\alpha > 0$  if the number  $2\beta$  is replaced by  $\gamma = \max(0, \beta, 2\beta + \alpha - 1)$ . All the other arguments in proving Theorem 1 for the general case remain the same with obvious changes.

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